Effective field theory approach for few-fermion systems in a trap *(for future applications to nuclear systems)*

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Ganil, January 11, 2010

nucleon-nucleon potential

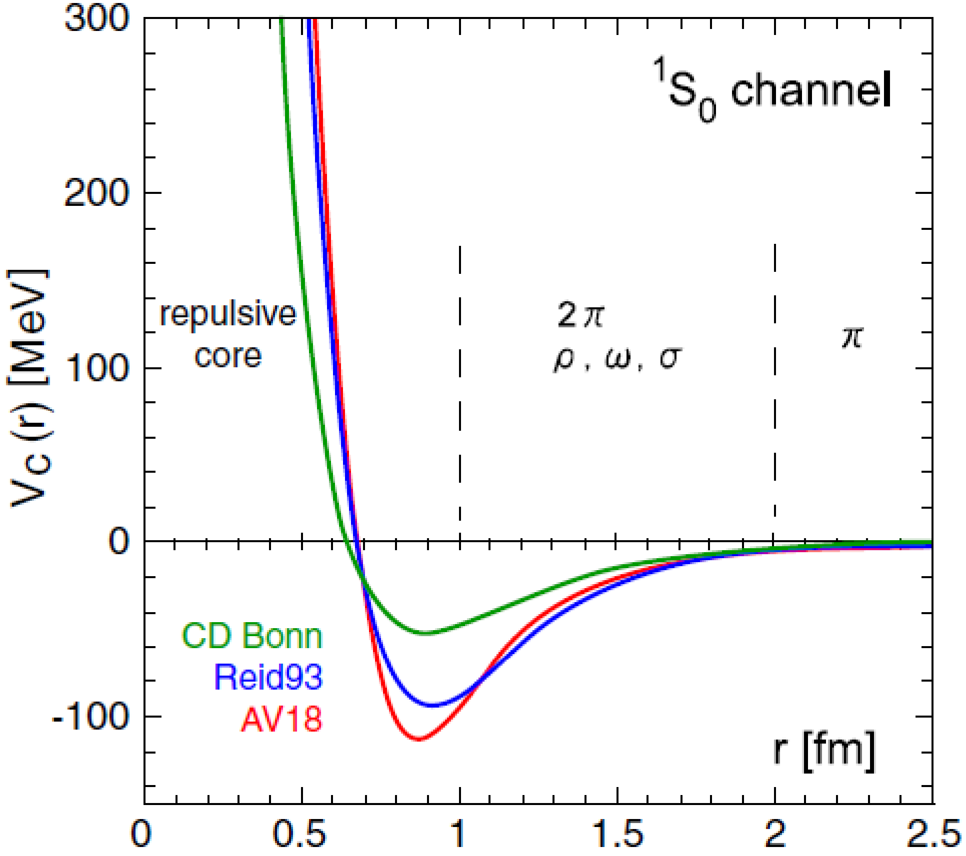


FIG. 1 (color online). Three examples of the modern *NN* potential in the $^1S_0$ (spin singlet and *s*-wave) channel: CD-Bonn [17], Reid93 [18], and AV18 [19] from the top at $r =$ 0.8 fm.

(taken from N. Ishii et al, PRL 99, 022001 (2007))

-> long range dominated by one-pion exchange.

-> medium range part receives contributions from the exchange of multipions and heavy mesons.

-> short range (r<= 1 fm) : strong repulsive core (quark-gluon structure of the nucleon).

-> two-nucleon data (phase shifts, deuteron binding energy) very well reproduced by realistic potential up to ~ 350 MeV, but...

Many-body nuclear systems :

i) problems associated with strong repulsive core at short distance (high energy)

-> possible solutions : softening of the core by using similarity transformation of the n-n potential (Lee-Suzuky transformation, Similarity Renormalization Group), v_{low_k} ....

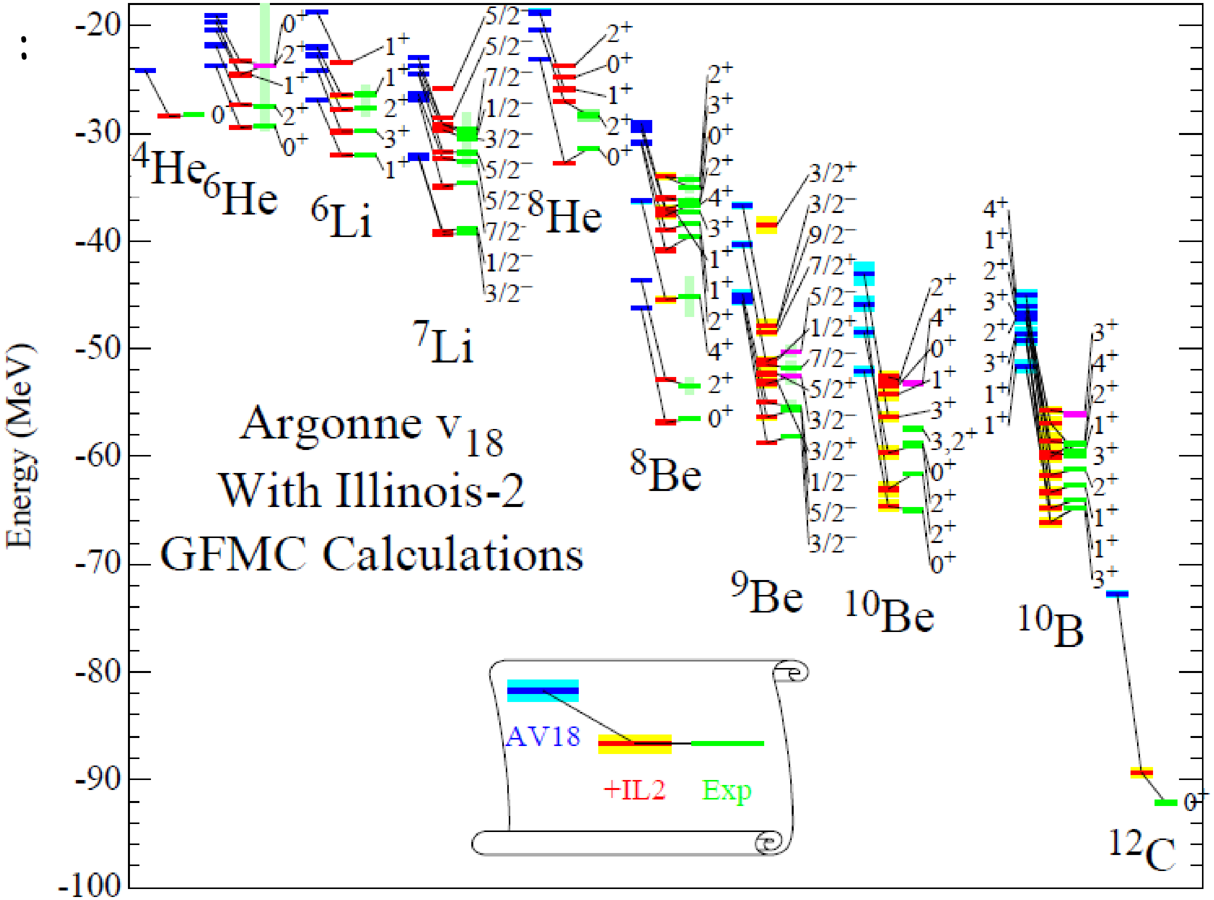
-> disadvantages : creation of many-body forces which also require intensive computational effort.

ii) "intrinsic" three-body force due to :

-> nucleon are not point particles (i.e not elementary)

-> some degrees of freedom are neglected e.g Δ -resonance, polarization effects.....

(taken from S. Pieper et al,
Nucl.Phys. A751 (2005) 516-532)

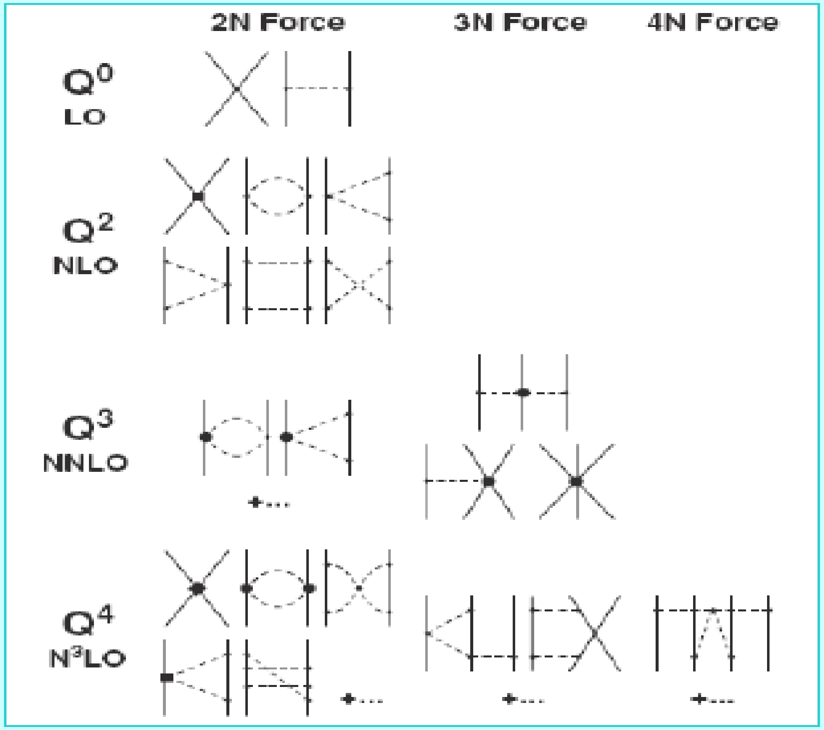


Effective Field Theory

S. Weinberg, Nucl. Phys. B363, 3 (1991).
U. van Kolck, Prog. Part. Nucl. Phys. 43, 337 (1999).

-> construction of soft interaction.
-> many-body and two-body interaction in a same framework.
-> improvable order by order.

Feynman diagrams



R. Machleidt and D. R. Entem, J. Phys. G 31 (2005) S1235

Effective field theory approach for few-fermion systems in a trap.

(for future applications to nuclear systems)

i) Why few-fermion systems in a trap ?

ii) Main ideas of Effective Field theory.

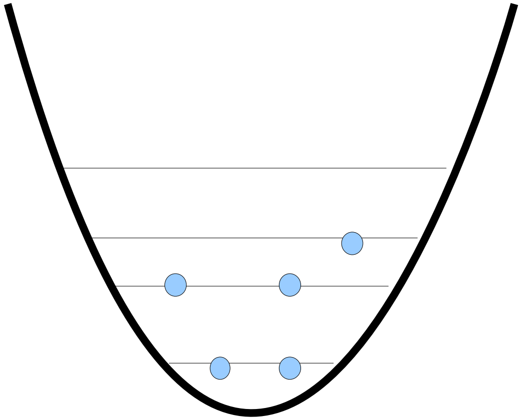
iii) Application to trapped systems.

iii) Conclusion and perspectives.

Ultra cold atoms confined in lattice by laser beam :

->interaction between atoms can be tuned with an external magnetic field.

-> experimental realization of the problem of trapped particles interacting via a zero range force.



Exact solutions are known for 2,3 particles in a trap.

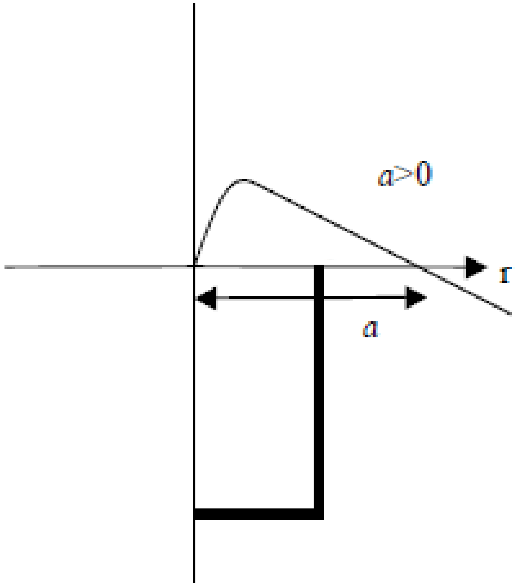
i) Nuclear Physics:

$a(^1S_0), a(^3S_1) >> 1/m_{\Pi} \sim 1.4 fm$

$a(^1S_0) \sim -20$ fm, $a(^3S_1) \sim 5$ fm

ii) Atomic physics:

-> ^{4}He atom systems with large scattering length.



→ Few atoms systems in a trap are a testing ground for few (many)-body methods.

Effective Field Theory (1/3)

i) Separation of scale :

M_{QCD} ~ 1 GeV (mass of nucleon)
M_{nucl} ~ 100 MeV (typical momentum in a nucleus)
M_{struct} ~ 10 MeV (binding energy of a nucleon in a nucleus)

-> details of physics at short distance (high energy) are irrelevant for low energy physics.

-> in EFT low energy degrees of freedom are explicitly included (high momenta are integrated out).

ii) The Lagrangian / potential consistent with symmetries is expanded as a Taylor Series:

$$V(\vec{p}\,', \vec{p}) = \sum_{i,j} C_{i,j} (\vec{p})^i (\vec{p}\,')^j$$

Effective Field Theory (2/3)

iii) Regularization and renormalization :

-> cut-off Λ (separation between low and high energy physics)

$$V(\vec{p}\,', \vec{p}) => \sum_{i,j} C_{i,j}(\Lambda)(\vec{p})^i(\vec{p}\,')^j$$

-> no dependence on cut-off for observables (for a high enough cut-off), dependence absorbed by coupling constants (fitted with observables).

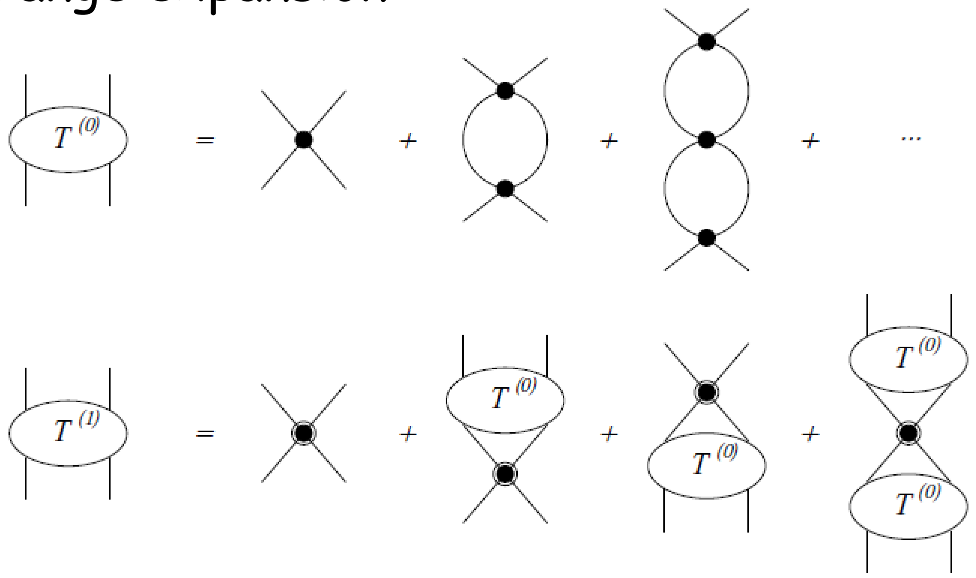
Effective Field Theory (3/3)

iv) Find the power counting ("truncation of the Taylor series"):

-> hierarchy between the different contributions
-> results improvable order by order (Leading Order, Next-to-Leading-Order, Next-to-Next-to-Leading-Order......)

-> how to find the power counting :

For instance in the continuum (no trap), comparison with effective range expansion :

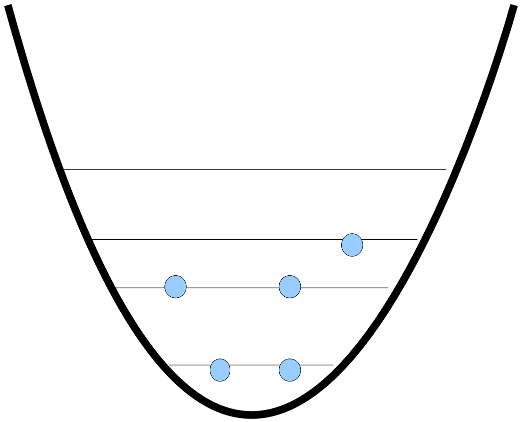


-> diagonalization at LO for system with large scattering length.
-> perturbation theory for higher orders.

(U. van Kolck, NPA, 645, (1999) 273)

Few-fermion system in a trap

$$H = \sum_{i<j} \frac{(\vec{p_i} - \vec{p_j})^2}{2mA} + \frac{1}{2}\frac{m\omega^2}{A}\sum_{i<j}(\vec{r_i} - \vec{r_j})^2 + \sum_{i<j} V_{ij}$$



How to solve the many-body equation ?

-> No Core Shell Model (P. Navratil, J. P. Vary, B.R Barrett, PRC 62, 054311 (2000))

-> expansion of the solution on a harmonic oscillator basis:

$$|\Psi\rangle = \sum_{n}^{n_{max}} a_n |SD_n\rangle$$

cutoff n_max <=> Λ

Two fermions in a trap.

Analytic solution for the two-fermion system given by:

$$\frac{\Gamma\left(\frac{3}{4}-\frac{\varepsilon}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{\varepsilon}{2}\right)} = -\frac{1}{2} bk \cot\delta$$

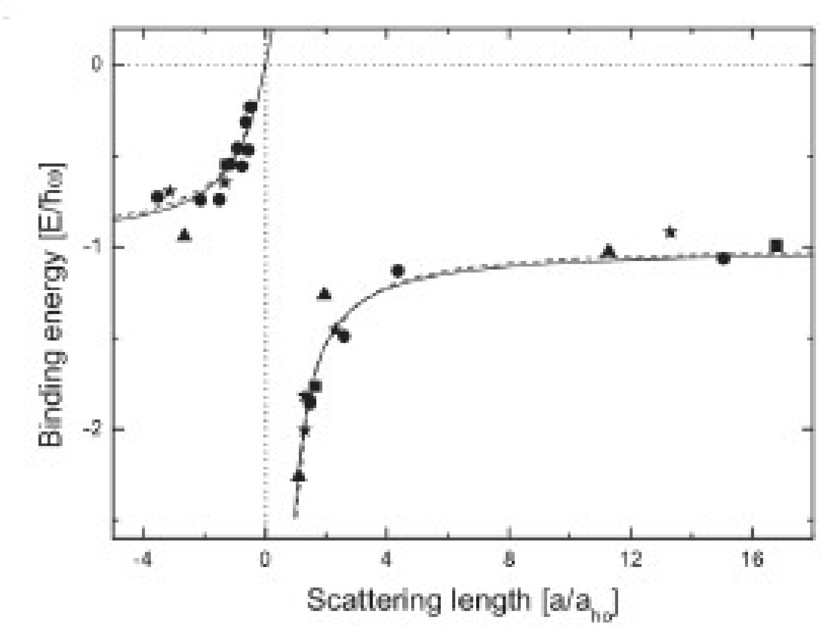
$$k\ \cot\delta = -\frac{1}{a_2} + \frac{1}{2} r_2 k^2 + \ldots,$$

T. Busch et. al., Found. Phys. **28** (1998) 549

Interaction (s-wave only) between fermions characterized by scattering length a_2, range r_2.......

$$b = \frac{1}{\sqrt{\mu\omega}}$$

$$\varepsilon = E/\omega$$



T. Stöferle et. al., Phys. Rev. Lett. **96** (2006) 030401

EFT approach : two-fermion in a trap at Leading Order (LO)

-> diagonalization of $V_{LO}(\vec{p},\vec{p}') = C_0$ in the truncated model space by a cutoff N_{max}

$$\psi(\vec{r}) = \sum_{n=0}^{N_{max}/2} A_n \phi_n(\vec{r})$$

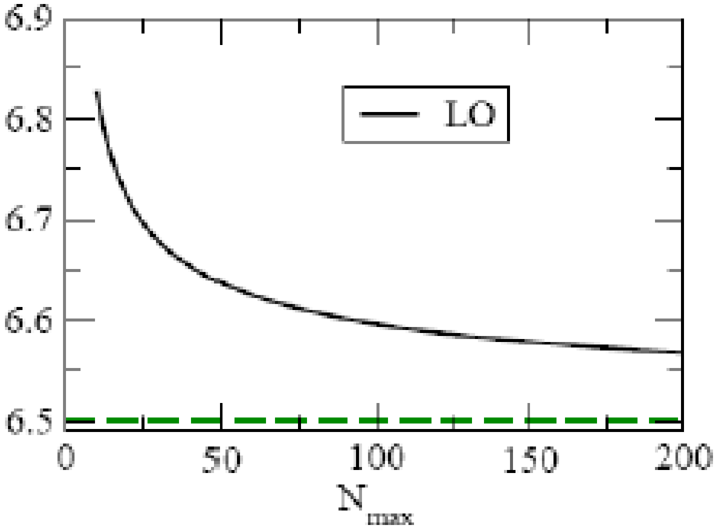
$$\left(\frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 r^2 + C_0(N_{max})\delta(\vec{r})\right)\psi(\vec{r}) = \varepsilon\omega\psi(\vec{r})$$

-> relation between the coupling constant C_0 and the energies:

$$\frac{1}{C_0(N_{max})} = -\sum_{n=0}^{N_{max}/2} \frac{|\phi_n(0)|^2}{2n+3/2-\varepsilon}$$

-> C_0 is fixed with one observable, others are predictions.

Third excited state at unitarity (infinite a_2)



-> NLO: first order perturbation theory

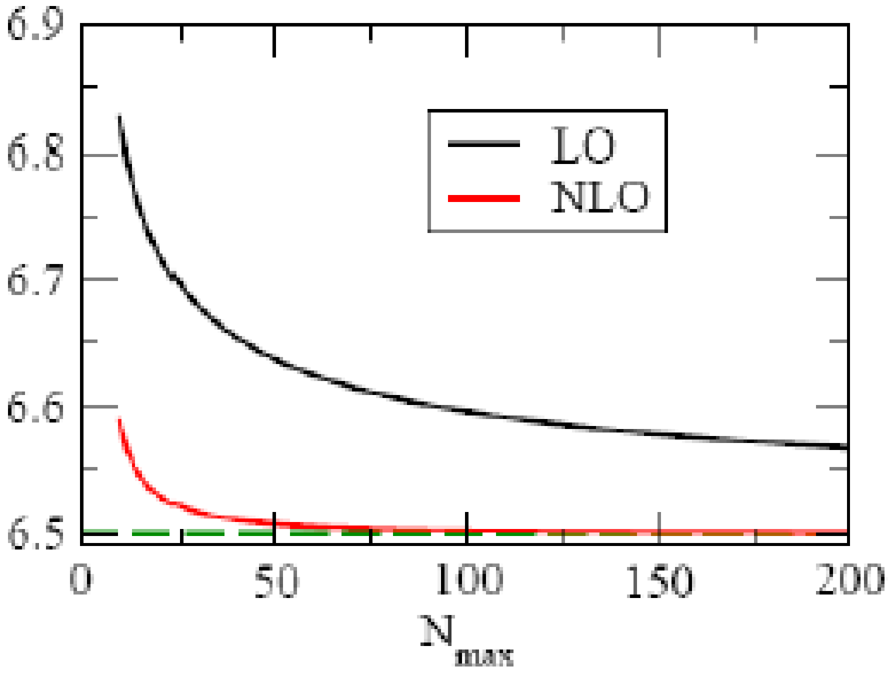
$$V_{NLO}(\vec{p}\,',\vec{p}) = C_2^{(1)}(\vec{p}\,'^2 + p^2)$$

$$\Delta\varepsilon_n = \langle\psi_n|V_{NLO}|\psi_n\rangle$$

$$\Delta\epsilon_n = \langle\Psi_n|C_2^{(1)}(p^2 + p\,'^2) + C_0^{(1)}|\Psi_n\rangle$$

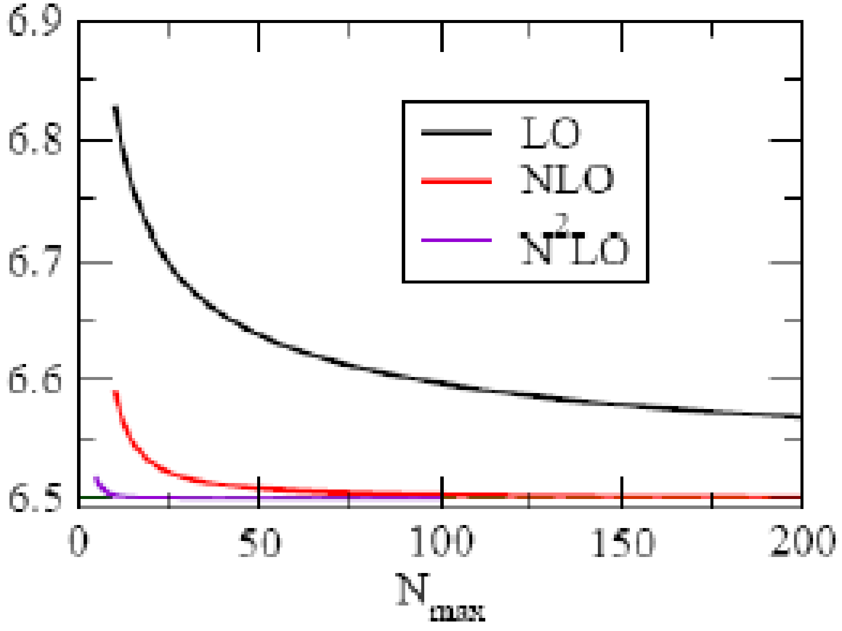
$$\Delta\varepsilon_0 = 0$$

$$\Delta\varepsilon_1 = \varepsilon_1^{exp} - \varepsilon_1^{LO}$$

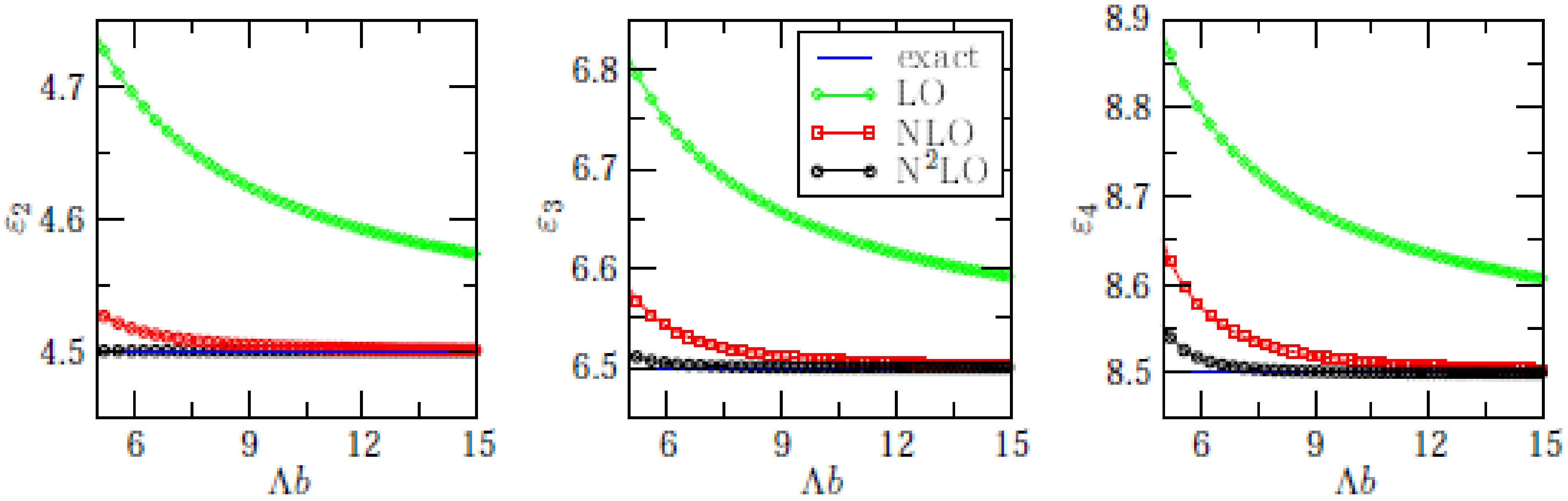


-> NNLO : 2nd order PT

$$\begin{aligned}\Delta\epsilon_n &= \sum_{i\neq n}\frac{|\langle\Psi_n|C_2^{(1)}(p^2+p\,'^2)+C_0^{(1)}|\Psi_i\rangle|^2}{\epsilon_i-\epsilon_n}\\ &+ \langle\Psi_n|C_2^{(2)}(p^2+p\,'^2)+C_0^{(2)}|\Psi_n\rangle\\ &+ \langle\Psi_n|C_4^{(2)}(p^2+p\,'^2)^2|\Psi_n\rangle\end{aligned}$$



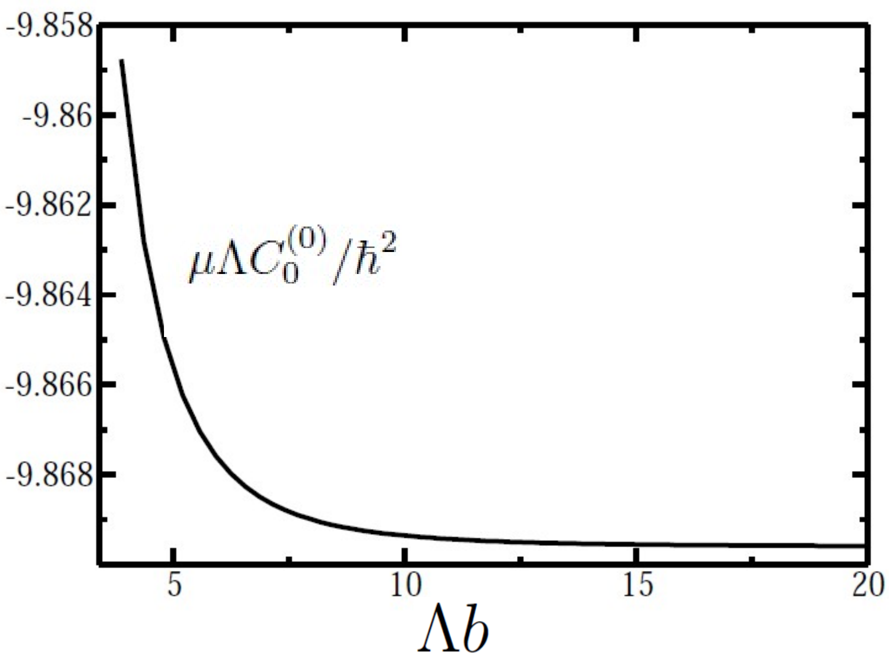
-> running of excited states at unitarity (infinite scattering length) :

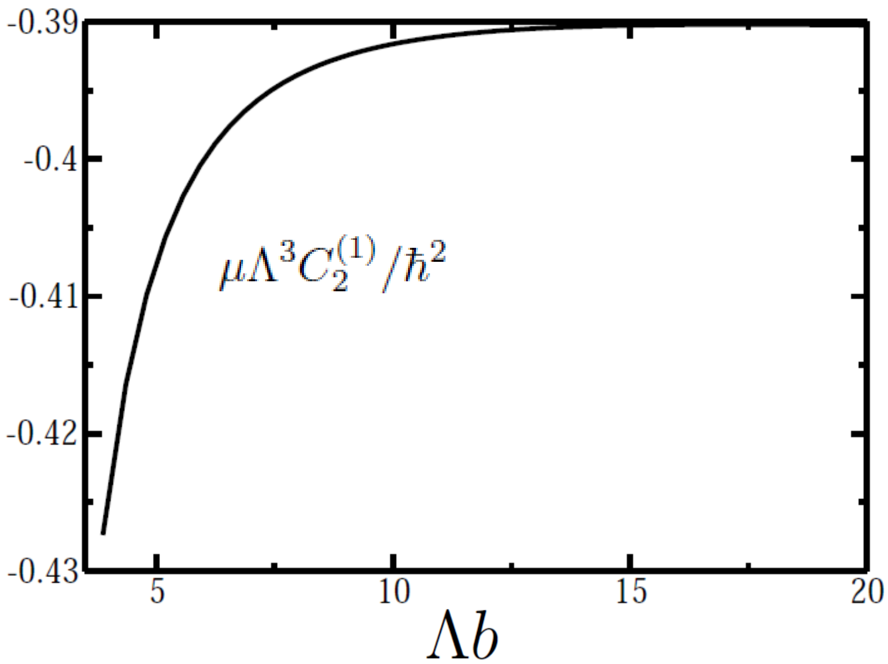


-> faster convergence (with the size of the model space) for low lying states but this is expected.

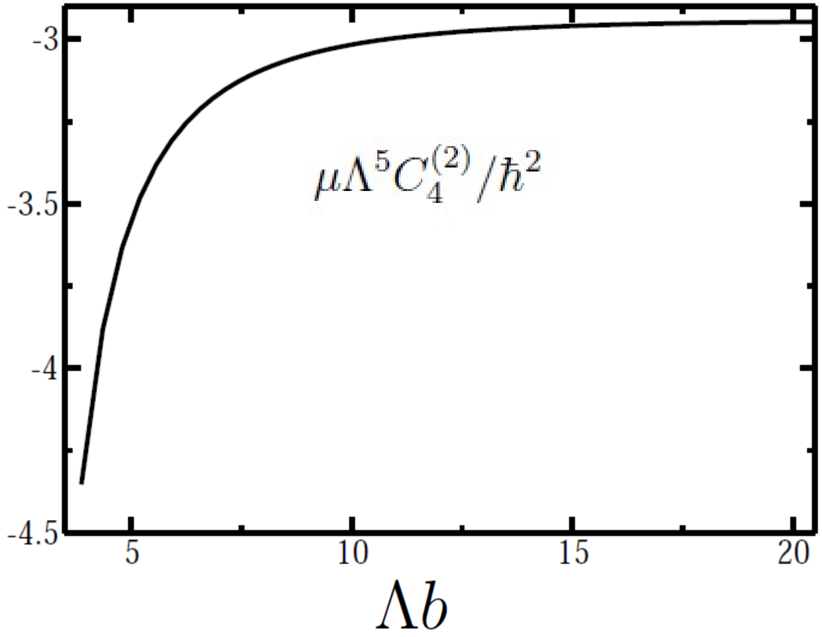
-> results improvable order by order.

Running of the coupling constants at unitarity.





$$
\begin{aligned}
V(\vec{p}\,',\vec{p}) &= \frac{1}{\Lambda}(\Lambda c_0(\Lambda)) \\
&+ \frac{1}{\Lambda}(\Lambda^3 c_2(\Lambda))\left[\left(\frac{\vec{p}\,'}{\Lambda}\right)^2+\left(\frac{\vec{p}}{\Lambda}\right)^2\right] \\
&+ \frac{1}{\Lambda}(\Lambda^5 c_4(\Lambda))\left[\left(\frac{\vec{p}\,'}{\Lambda}\right)^4+\left(\frac{\vec{p}}{\Lambda}\right)^4\right]
\end{aligned}
$$



-> good power counting !

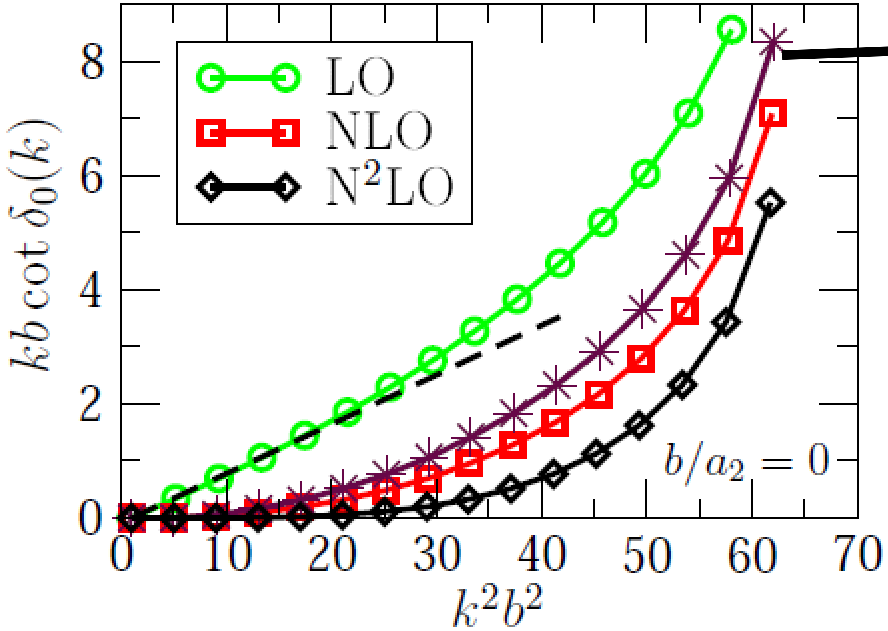
Solution at unitarity:

exact

$$\frac{\Gamma\left(\frac{3}{4}-\frac{\varepsilon}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{\varepsilon}{2}\right)} = -\frac{1}{2} bk \cot\delta_0(k) = 0$$

EFT in truncated space

$$\frac{\Gamma\left(\frac{3}{4}-\frac{\varepsilon_n}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{\varepsilon_n}{2}\right)} = -\frac{1}{2} bk_n \cot\delta_0(k_n)$$



"full" diagonalization of

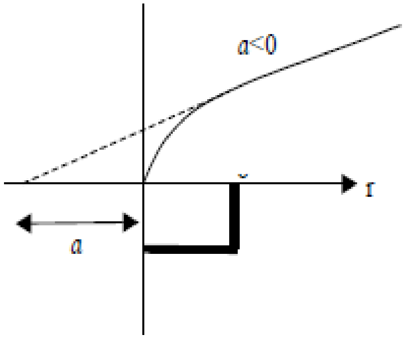
$$V(\vec{p}\,',\vec{p}) = c_0 + c_2\left(\vec{p}\,'^2 + \vec{p}^{\,2}\right)$$

-> worse than perturbations only (for higher orders than LO).

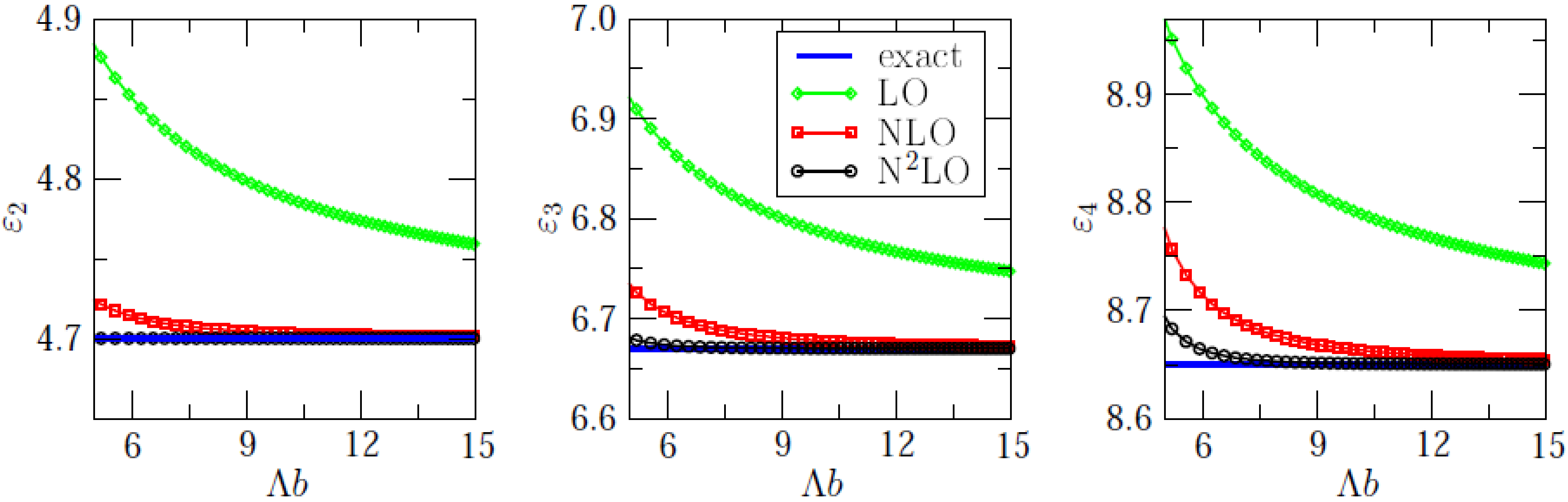
-> another verification of the power counting.

I.Stetcu, J.R, B. Barrett, U. Van Kolck, in preparation

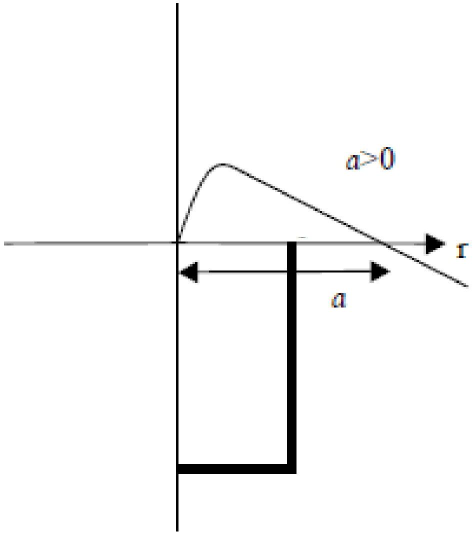
Weak coupling



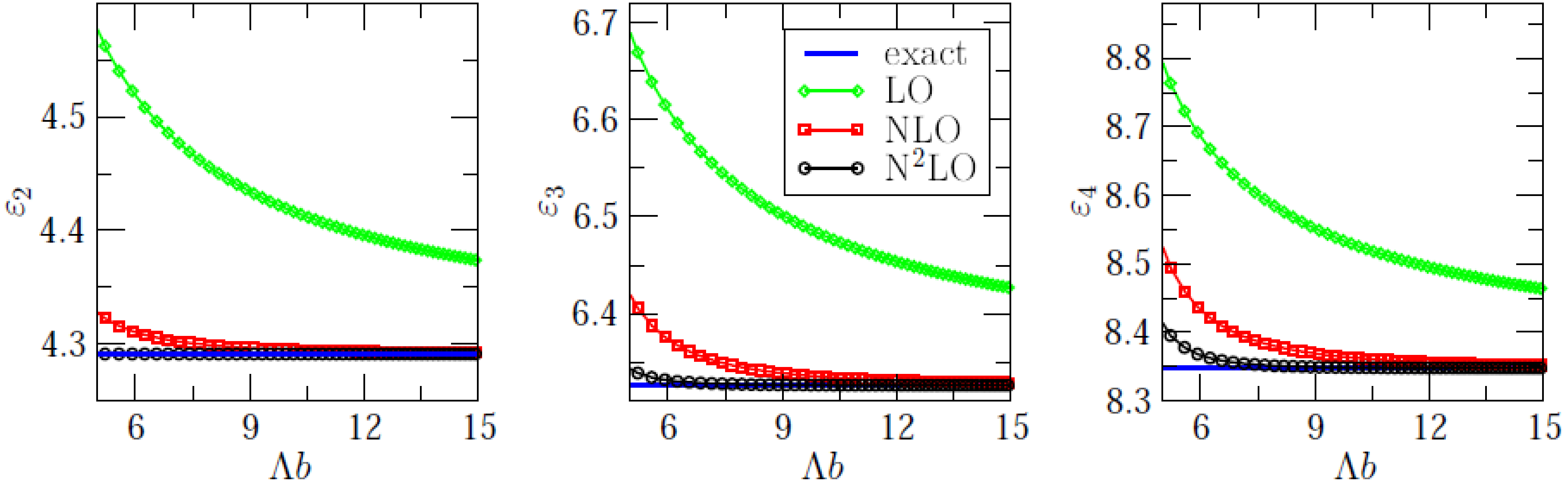
-> b/a=-1



Stronger coupling

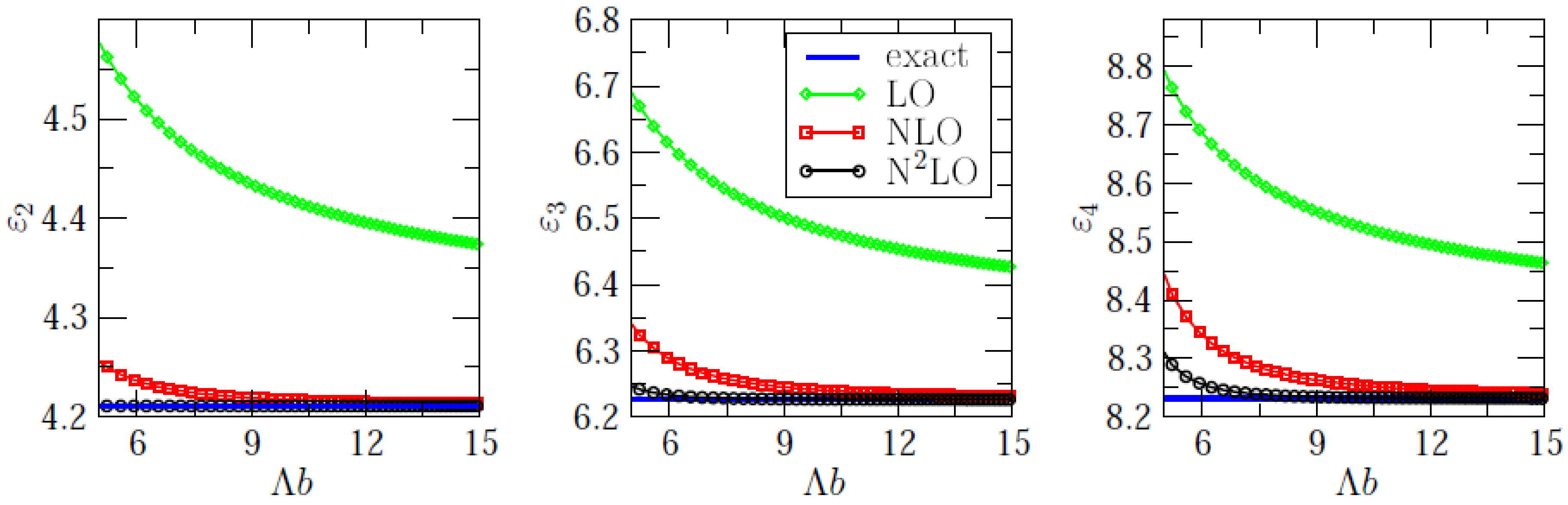


-> b/a =1



-> interaction with a finite range : b/a=1, r/b=0.1

$$\frac{\Gamma\left(\frac{3}{4}-\frac{\varepsilon}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{\varepsilon}{2}\right)} = -\frac{1}{2}bk\cot\delta = \frac{b}{2a_2} - \frac{r_2bk^2}{4}$$



-> qualitatively no change with respect to zero range case

Few-body problem:

-> how large are the many-body forces ?

$$H_{int} = \sum_{i=1} \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 r_i^2 - H_{CM} + \sum_{i,j} V(i,j) + \sum_{i,j,k} V(i,j,k) + \dots$$

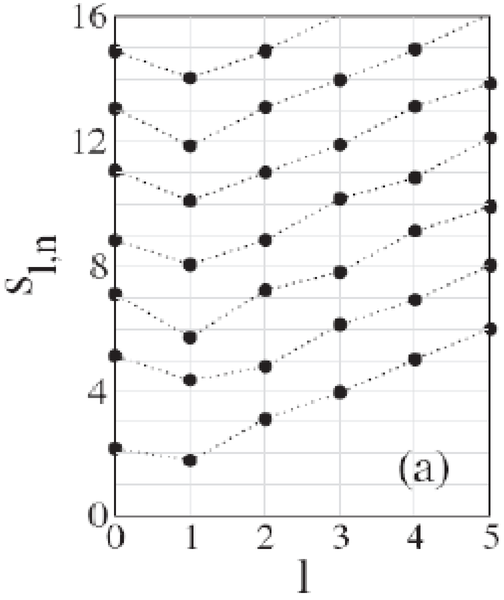
-> relation between two-body and many-body cut-off

Three fermions in a trap at unitarity :

-> Free Schrodinger Equation with boundary condition :

$$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \left(\frac{1}{r_{ij}} - \frac{1}{a_2} \right) A(\vec{R}_{ij}, r_k) + \mathcal{O}(r_{ij})$$

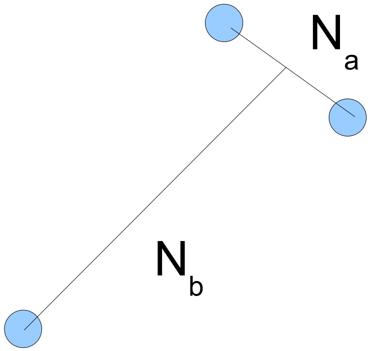
$$E = E_{c.m.} + (s_{l,n} + 1 + 2q)\omega$$



F. Werner and Y. Castin, Phys. Rev. Lett. **97**, 150401 (2006)

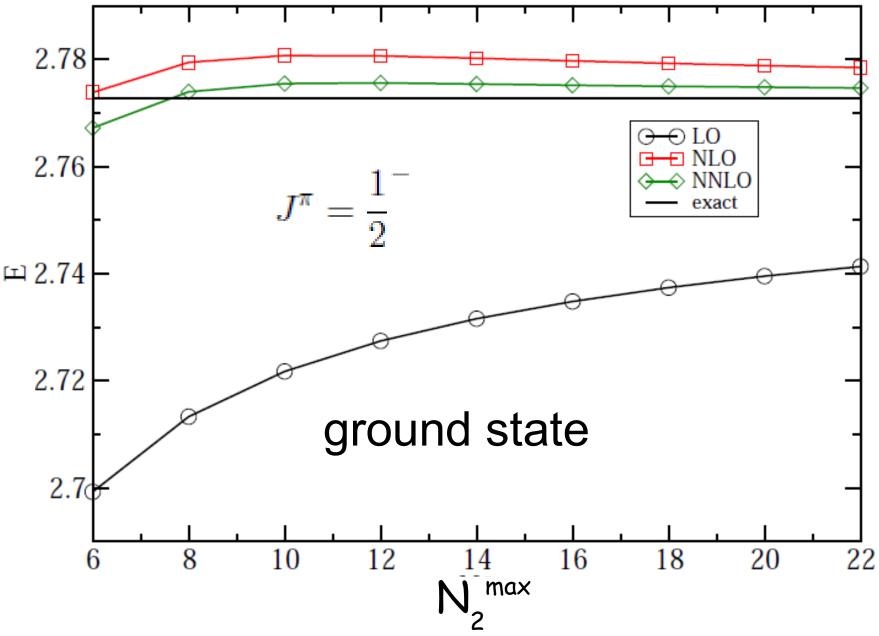
Three fermions in a trap

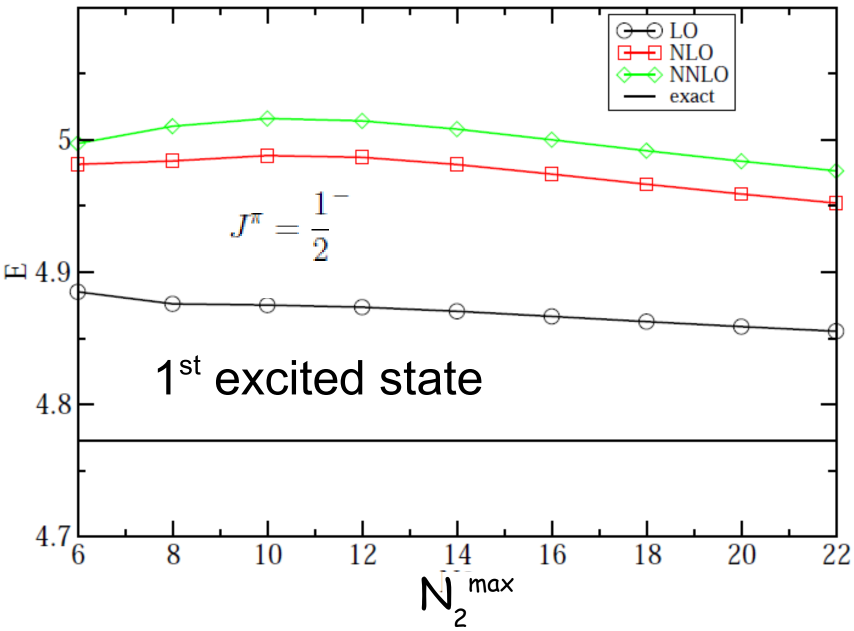
-> two-body interaction defined by cutoff N_2^{max} and three body model space defined by cutoff N_3^{max}



N_3=Na+Nb

A) cut off *a la* Shell Model : two body cut-off N2 is fixed by N3.



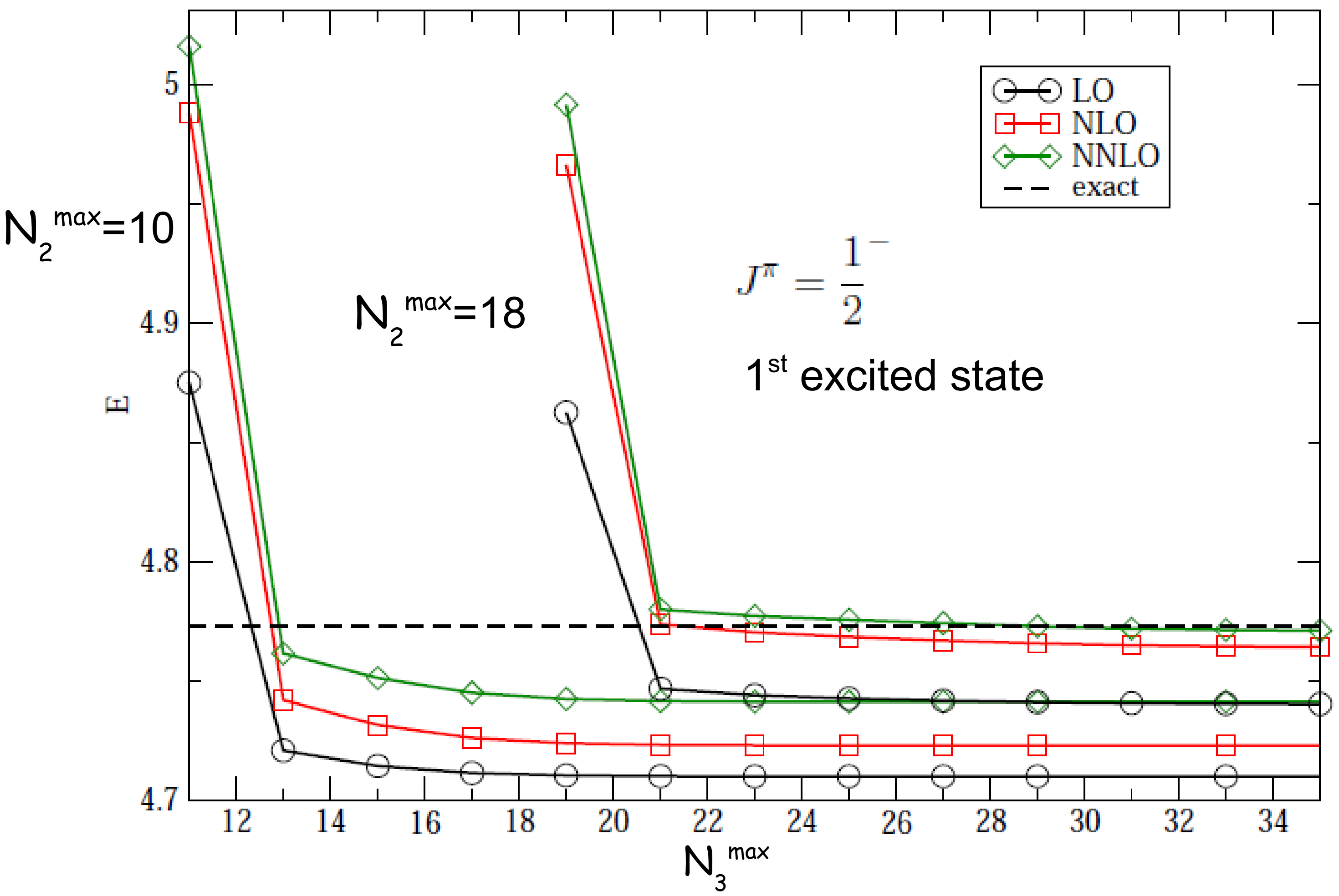


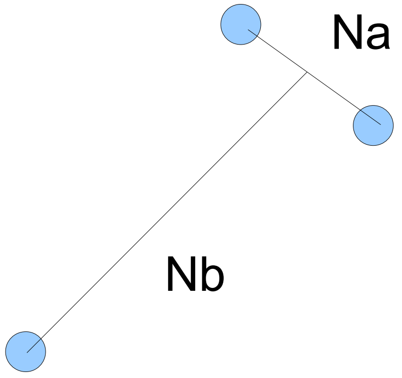
→ Problem !!! : NLO , NNLO further away from exact value than LO for some states

Three fermions in a trap

B) Solution :

-> for a fixed two-body cut-off N_2^{max} the three body cut-off N_3^{max} is increased until convergence (completeness is reached).





N3=Na+Nb

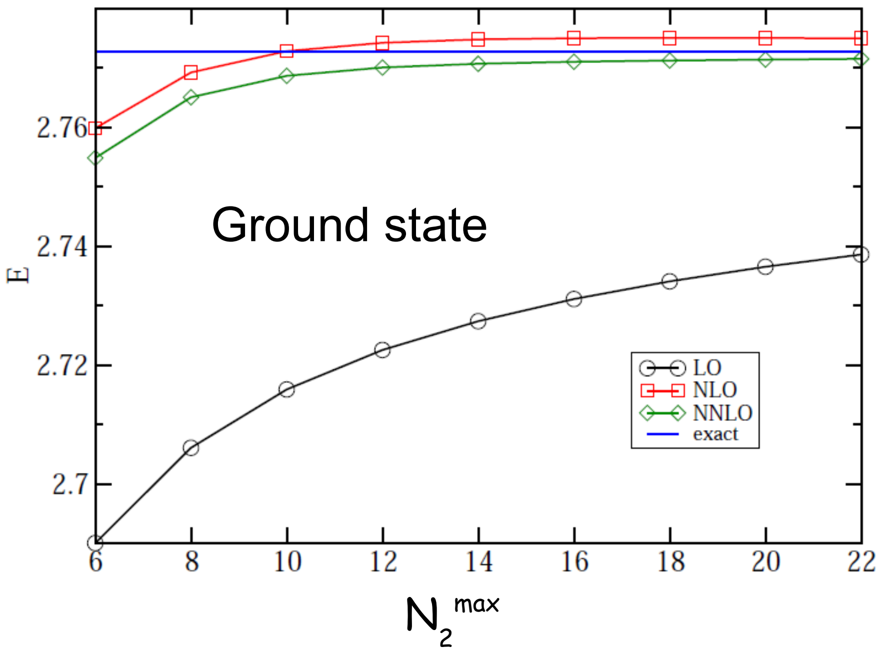
->no restriction on N3

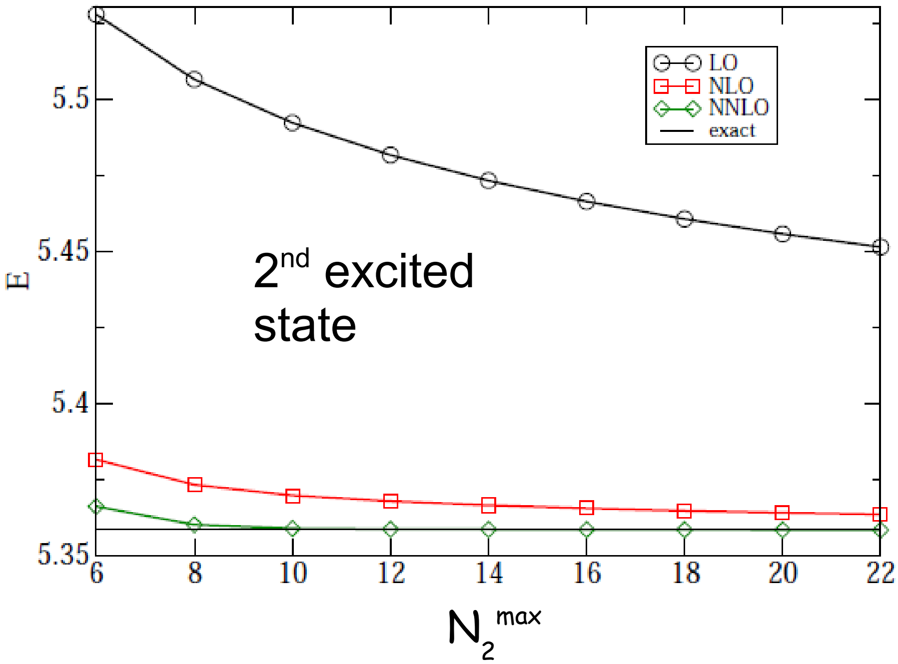
-> for $N_a > N_2^{max}$ the interaction is "switched off"

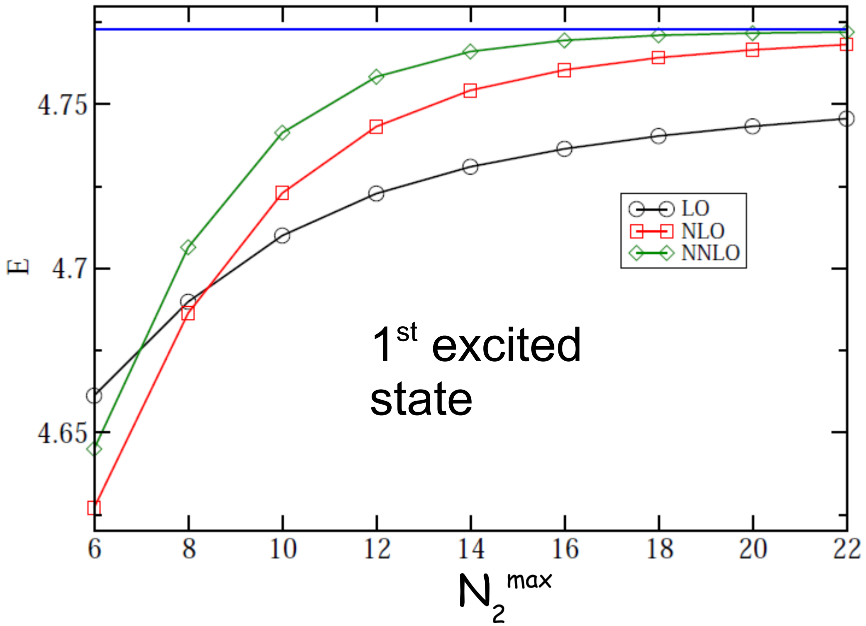
-> correct ordering of the different orders, faster convergence

J. R, I.Stetcu, B. Barrett, U. Van Kolck, M. Birse, in preparation.

$$J^\pi = \frac{1}{2}^-$$







-> convergence increases as more corrections are considered

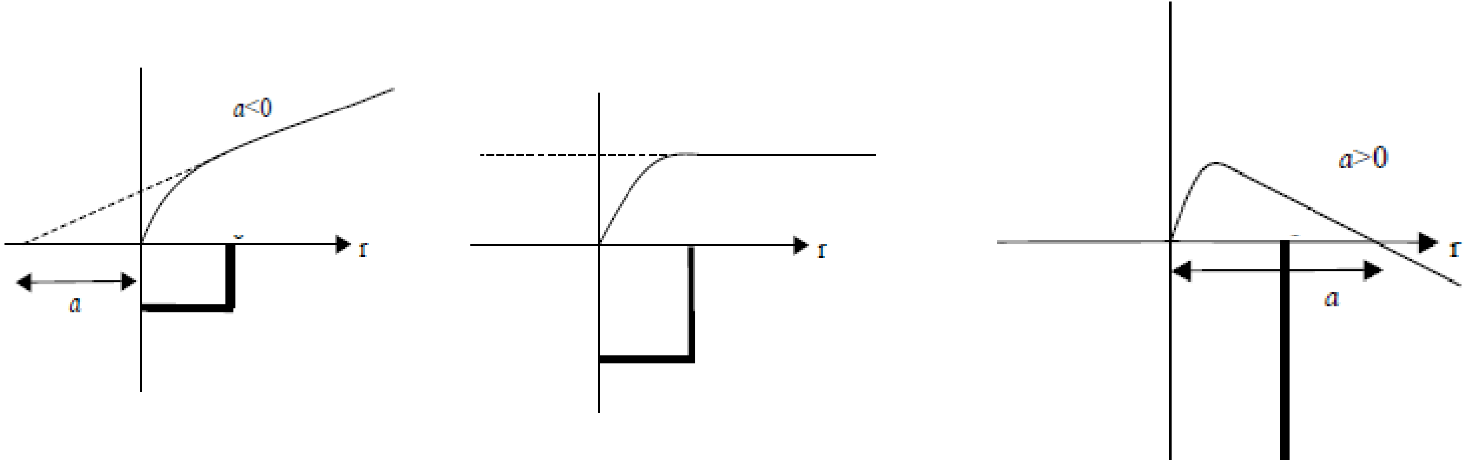
->three-body force not necessary (for this system)

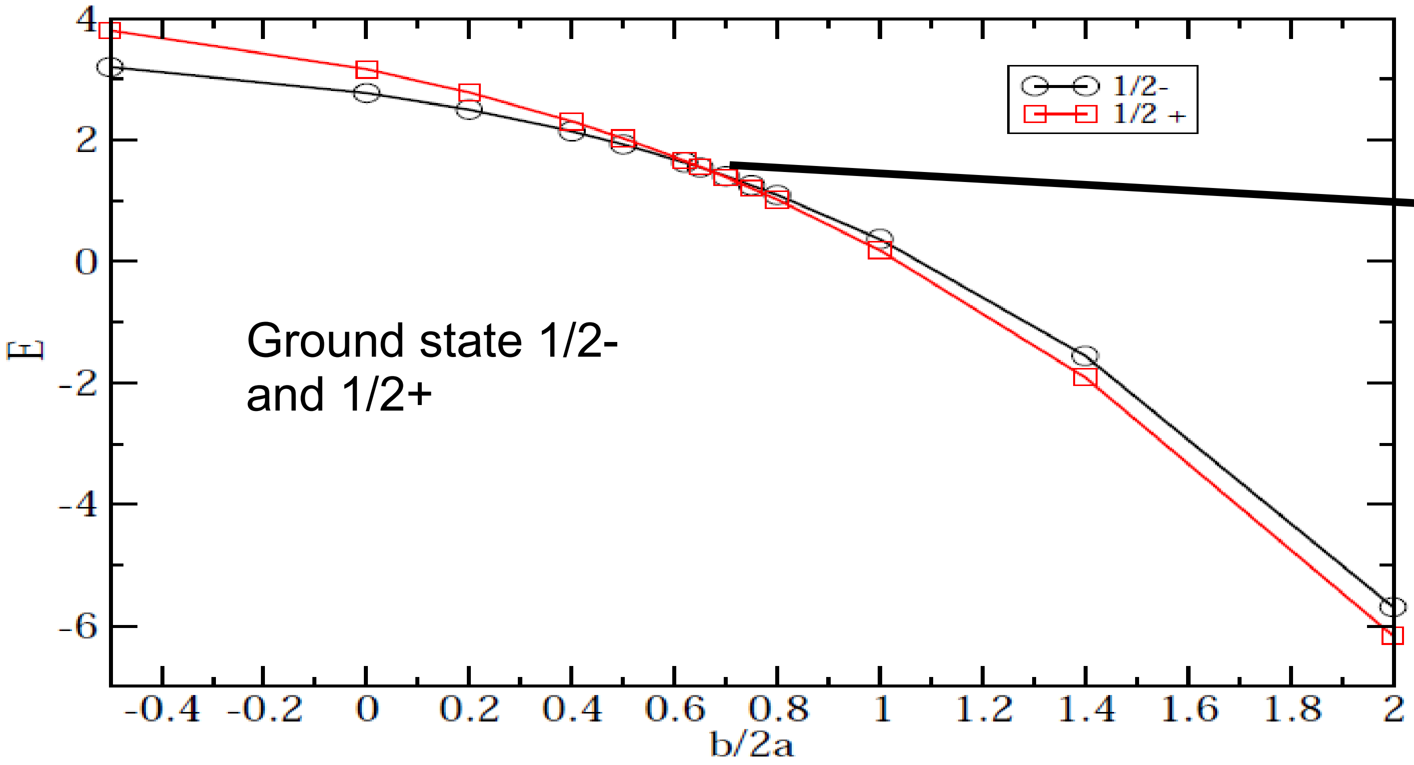
Evolution of the ground state of the 3-fermion system with b/2a

weak coupling

unitarity

stronger coupling

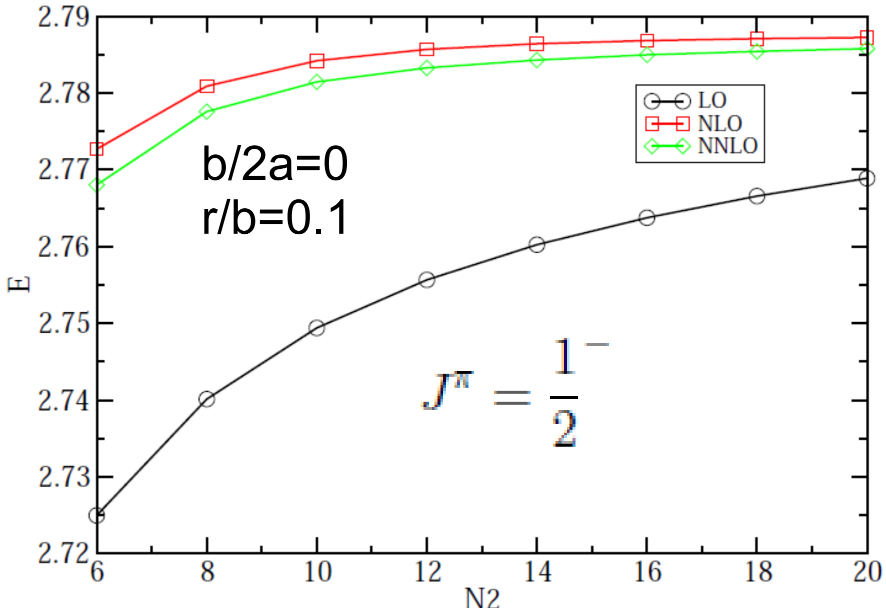


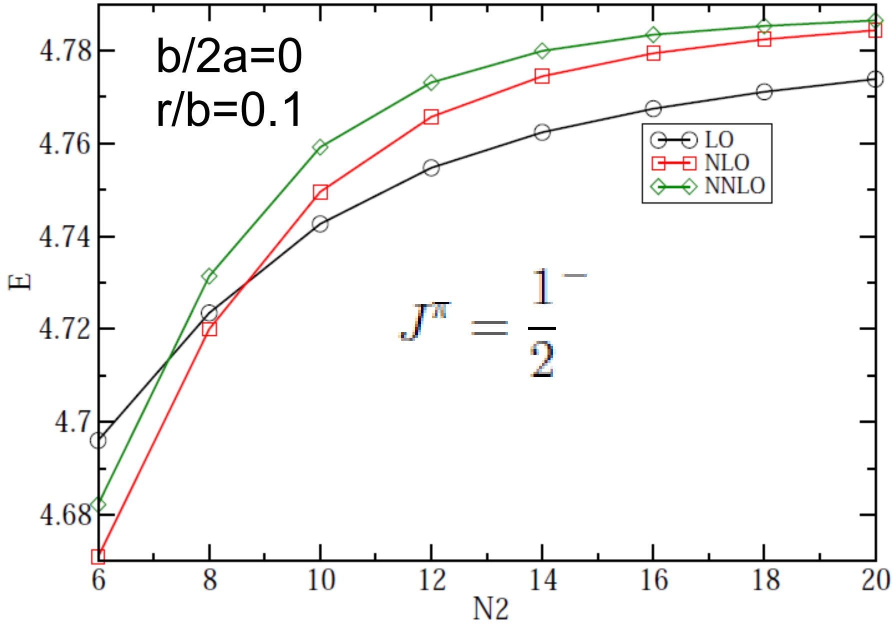


Ground state 1/2- and 1/2+

Parity inversion of the ground state

Three-fermion system with infinite two-body scattering length and finite two-body range.





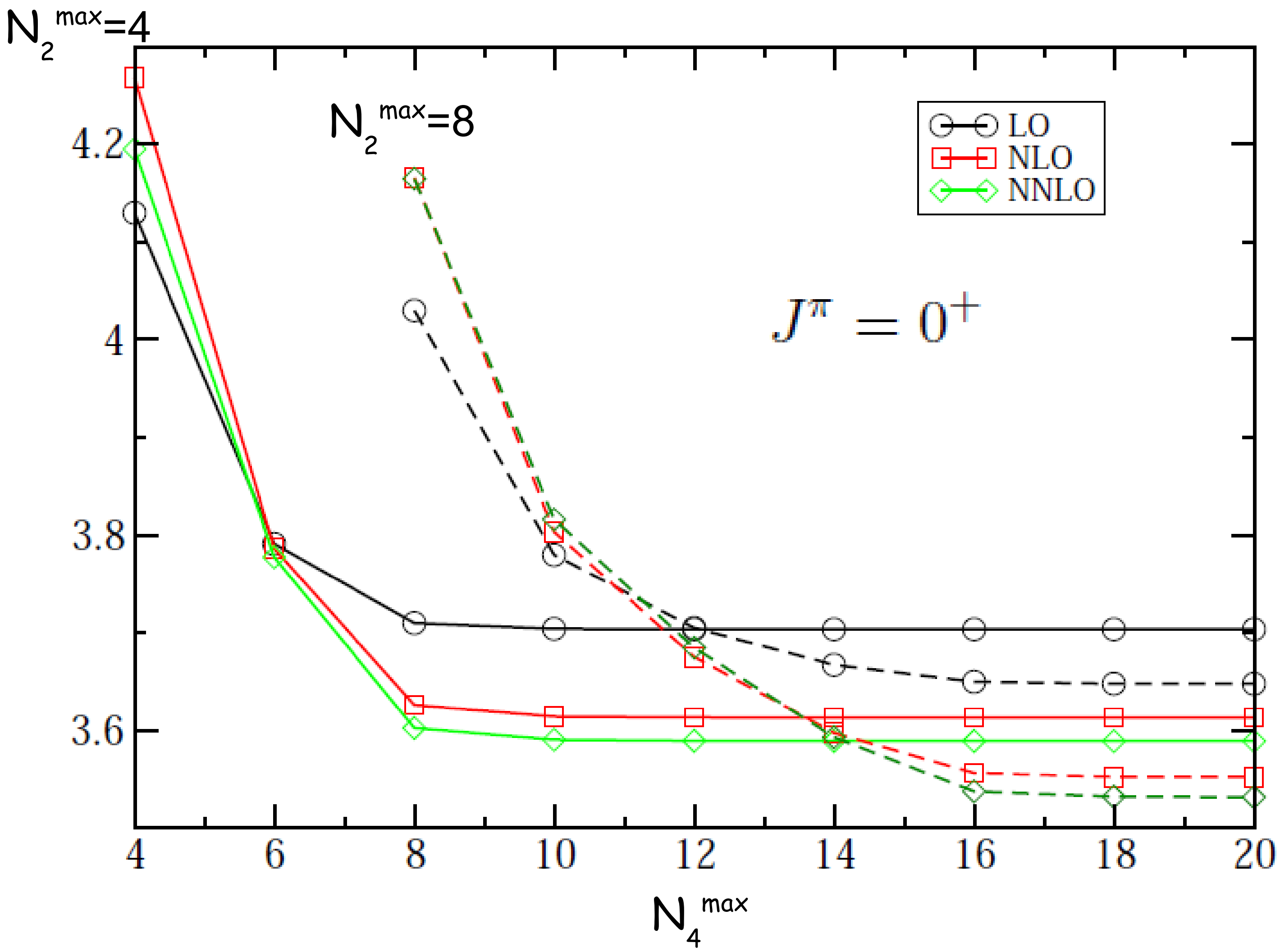
Two-body system :

$$\frac{\Gamma(3/4-\varepsilon/2)}{\Gamma(1/4-\varepsilon/2)} = -\frac{b}{2}\left(-\frac{1}{a_2} + r_0\varepsilon\right)$$

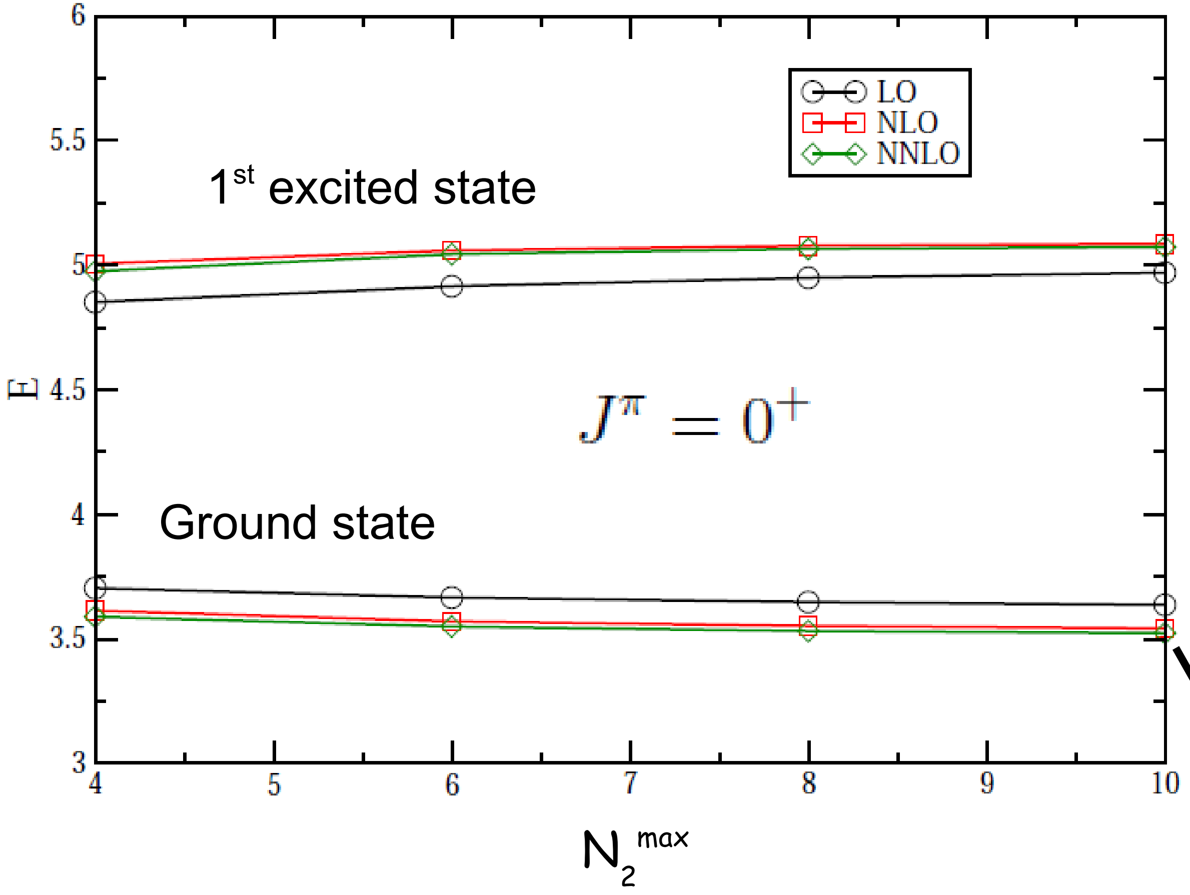
→ fixes the coupling constants

Four fermion system at unitarity

-> N_2^{max} fixed, N_4^{max} increased until four body completeness reached



Four fermion system at unitarity



NNLO, N_2^{max}=10

E= 3.52

Comparison with other approaches:

-> 3.545 ∓0.003
Alhassid et al, PRL 100 230401 (2008)

-> 3.6 ∓0.1
Chang et al, PRA 76, 021603 (2007)

Conclusions

-> Application of Effective Field Theory for few-fermion system in a trap.
-> potential is expanded as a Taylor Series and results are improvable order by order
-> smooth interaction but no similarity transformation
-> results for 3-body -> excellent agreement with exact solution.
-> 4-body: good agreement with other approaches.

Application to Nuclear physics :

Construction of nuclear interaction with Effective Field Theory approach:

-> two-body and many- body force in a same framework
-> use of the trap to built a bridge between
the scattering properties of the interaction (future results from lattice QCD calculations) and the bound state physics (Shell Model).

-> application for nuclei far away from stability :
- * halo systems (radial extension >> range of nuclear interaction)
- * weakly-bound bound / unbound systems within a (No-core) Gamow Shell Model approach.